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Fibonacci, Van der Corput and Riesz–Nágy

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ABSTRACT

What do the three names in the title have in common? The purpose of this paper is to relate them in a new and, hopefully, interesting way. Starting with the Fibonacci numeration system — also known as Zeckendorff's system — we will pose ourselves the problem of extending it in a natural way to represent all real numbers in $(0,1)$. We will see that this natural extension leads to what is known as the ϕ-system restricted to the unit interval. The resulting *complete system of numeration* replicates the uniqueness of the binary system which, in our opinion, is responsible for the possibility of defining the Van der Corput sequence in $(0,1)$, a very special sequence which besides being uniformly distributed has one of the lowest discrepancy, a measure of the goodness of the uniformity. Lastly, combining the Fibonacci system and the binary in a very special way we will obtain a singular function, more specifically, the inverse of one of the family of Riesz–Nágy.

1. Introduction

We are interested in two problems, apparently different: the unique representation of positive integers as a sum of a fixed set of natural numbers using what is usually called a *system of numeration*, and the problem of representing uniquely a real number, $0 < x < 1$, as the sum of a series of positive terms, which is usually referred to as a *system of representation*. When these two systems — that is, both sums — can be expressed in the same way as a unified two-way sequence we may say we have a *complete system of numeration*. This is the case for the usual decimal system. When generalized to a given base, an integer $b \geqslant 2$, any $x \in \mathbb{R}^+$ can be written as

$$x = \sum_{i=-\infty}^{+\infty} a_i \cdot b^i, \quad a_i \in \mathbb{Z};\ 0 \leqslant a_i \leqslant b-1, \tag{1}$$

with finitely many $a_i \neq 0$ for $i \geqslant 0$. The a_i are called the *digits* of the system and the system is called positional in the sense that writing only the sequence $\langle a_i \rangle$, the position of each digit determines completely the number represented. We usually write

$$x = a_n a_{n-1} \ldots a_1 a_0 . a_{-1} a_{-2} \ldots.$$

A very interesting peculiarity of the binary system is that if you write the sequence of all natural numbers in order:

$$1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, \ldots$$

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and you obtain the mirror image of this sequence with respect to the "decimal point":

$$0.1, 0.01, 0.001, 0.011, 0.111, 0.0001, 0.1001, 0.0101, \ldots \tag{2}$$

you get a sequence known as the Van der Corput's sequence [1]. The interesting thing about this sequence is that it is uniformly distributed in $(0, 1)$. We believe that this is a consequence of the *good behavior* of the binary system both as a system of numeration and as system of representation.

This is not always the case. There are many systems of representation of real numbers in $(0, 1)$ that have no counterparts for the positive integers. We mention a few that have had their importance in the literature: Oppenheim series [2], (α, γ)-expansions [3], β-expansions [4,5], Lüroth series [6], Engel's series [7], Sylvester's series [8], $(\tau, \tau - 1)$-expansions [9], and a long list that, unavoidably, contains repetitions as different authors rediscover the same developments or generalize others already known. The best reference for many of these systems of representation is [10] and the bibliography therein. We have excluded representations using negative terms and other "exotic" ways of developing a real number which can be also found in [10]. Just to make clear what we mean by "exotic representation systems" we mention a few: besides the well-known regular continued fractions there are different ways of expanding real numbers as other types of continued fraction [11], or as f-expansions [4,12], or infinite products [13–15].

In this paper we will be interested in *binary systems*, that is, systems in which the digits, the a_i in the sums (1), can only be 0 or 1. There are an infinity of these, surprising as it may seem, and as far as we know, only a few have a counterpart as a coherent system of representation, coherent in the sense of the good distribution of the specular image of the sequence $\langle 1, 2, 3, 4, 5, \ldots\rangle$ expanded in the corresponding system of representation.

One of these *well-behaved* systems is the Fibonacci numeration system. Its counterpart as a representation system will also be related to Fibonacci: it will be a series expansion based on the powers of the reciprocal of the golden mean, ϕ. We will see that the Van der Corput's sequence of this system is also uniformly distributed in $(0, 1)$.

Lastly, using both the usual binary system and the Fibonacci system we will construct a function $L : [0, 1] \to [0, 1]$, continuous, strictly increasing and whose derivative is 0 almost everywhere in $[0, 1]$. These functions are known as singular. A typical example of a class of singular functions was provided by Riesz and Nágy [16]. Surprisingly enough, the inverse of our function coincides with one of the Riesz and Nágy class.

A good description of the different positional systems of numeration can be found in [17] but, as we are only interested in those that only use 0 and 1 as digits we will restrict the general setting found there.

2. Binary systems of numeration

Let $1 = L_0 < L_1 < L_2 < L_3 < \cdots$ be a strictly increasing sequence of nonnegative integers. Our aim is to represent uniquely any positive integer, N, as a sum of different terms L_i. If we succeed we will have

$$N = L_{i_1} + L_{i_2} + \cdots + L_{i_j}, \quad i_1 > i_2 > \cdots > i_j, \tag{3}$$

and we will be able to represent the sum above positionally as a sequence of 0 and 1: 1 in place $m + 1$ if L_m appears in (3) and 0 otherwise. For instance, $N = L_7 + L_5 + L_4 + L_2$, will be written

$$N = 10110100.$$

The sequence $S = \langle L_i\rangle$ will be called the *base* of the *binary system of numeration* (BSN).

As we will see, not all sequences $1 = L_0 < L_1 < L_2 < L_3 < \cdots$ will be adequate for our purpose. In order to obtain a BSN, there are conditions to be imposed to the L_i. Let us find them.

It is obvious that the sequence $\langle L_i\rangle$ partitions the set of all integers greater or equal than one:

$$[1, \infty) = \bigcup_{i=0}^{\infty} [L_i, L_{i+1}).$$

Now, in order to obtain the desired expansion (3), we apply to a positive integer, $N \geqslant 1$, the following greedy algorithm:

Let $N = N_1$. We define inductively the positive integer $N_k > 0$ $(k \geqslant 1)$ as follows: let i_k be such that

$$N_k \in [L_{i_k}, L_{i_k+1}).$$

We define N_{k+1} as follows:

$$N_k = L_{i_k} + N_{k+1}, \quad \text{where } N_{k+1} \in [0, L_{i_k+1} - L_{i_k}). \tag{4}$$

If $N_{k+1} = 0$ the algorithm terminates. If not, we iterate the process till we reach a *residue*, $N_{i_j} = 0$ that terminates the algorithm. The algorithm must eventually terminate because otherwise we would get an integer as a sum of an infinity of positive integers. This algorithm will provide the representation (3) of N in terms of the elements in $\langle L_i\rangle$.

In order to obtain a BSN, we need to impose some conditions to expansion (3) above:

(a) Any term L_i must appear only once.
(b) The expansion has to be [almost] unique.

Condition (a) is self-explanatory as we require that only 0 and 1 be possible digits. It is also partly related to the uniqueness of the expansion (clearly, any positive integer N can be always be written as $N = 1 + \overset{N)}{\cdots} + 1$). Lastly, condition (a) implies that $L_1 = 2$. If not, 2 would have no expansion with different L_i. Similarly L_2 can only be 3 or 4. In this way, we see that the growth of the L_i has to be a very controlled one. A little reflection shows that to ensure condition (a) it is necessary that

$$L_{i+1} - L_i \leqslant L_i \quad \forall i \in \mathbb{Z}^+. \tag{5}$$

This is also sufficient as it ensures that the residue in algorithm (4) verifies $0 \leqslant N_{j+1} < L_{i_j}$ and, consequently, L_{i_j} does not enter again the sum.

As for condition (b), the "almost" in it needs an explanation. If we wish the strict fulfillment of condition (b), that is, the uniqueness of the expansion, the possible BSN get quite restricted. The following lemma is quite illuminating:

Lemma 2.1. *Under the BSN with base $S = \langle L_i \rangle$, a positive integer has a unique expansion of the form*

$$L_{i_1} + L_{i_2} + \cdots + L_{i_j} \quad \text{with } i_1 > i_2 > \cdots > i_j$$

if and only if

$$L_{i+1} - 1 = L_i + L_{i-1} + \cdots + L_0 \quad \forall i \in \mathbb{Z}^+. \tag{6}$$

Proof. [$\Rightarrow$] If for a given i we had $L_{i+1} - 1 > L_i + L_{i-1} + \cdots + L_0$, then, clearly enough, the algorithm applied to the number $N := L_i + L_{i-1} + \cdots + L_0 + 1$ would output an expansion with L_0 repeated (recall that $L_0 = 1$). It is clear, then, that

$$L_{i+1} - 1 \leqslant L_i + L_{i-1} + \cdots + L_0 \quad \forall i \in \mathbb{Z}^+. \tag{7}$$

Now, if for a given i we had $L_{i+1} - 1 < L_i + L_{i-1} + \cdots + L_0$ we would have $L_{i+1} \leqslant L_i + L_{i-1} + \cdots + L_0 =: N$ and the expansion of this N provided by the algorithm would output in the first place an L_m larger or equal than L_{i+1}, when it is obvious that the algorithm applied to N must output $N = L_i + L_{i-1} + \cdots + L_0$. We have thus proved that

$$L_{i+1} - 1 \geqslant L_i + L_{i-1} + \cdots + L_0 \quad \forall i \in \mathbb{Z}^+. \tag{8}$$

Both (7) and (8) prove (6).

[$\Leftarrow$] If there existed an N with two different expansions:

$$N = L_{i_1} + L_{i_2} + \cdots + L_{i_j} = L_{k_1} + L_{k_2} + \cdots + L_{k_s}$$

with $i_1 > i_2 > \cdots > i_j$ and $k_1 > k_2 > \cdots > k_s$, after canceling all equal terms on each side the equation, we would end up with a similar expression

$$L_{i_h} + \cdots = L_{k_t} + \cdots \tag{9}$$

where, let us say, $L_{i_h} > L_{k_t}$. Writing

$$L_{i_h} = L_{i_h - 1} + L_{i_h - 2} + \cdots + L_0 + 1$$

all terms on the right-hand side of (9) would vanish as they canceled with the corresponding terms on the left-hand side. This would lead to

$$L_{i_h} + \cdots = 0$$

which is clearly impossible. $\square$

Condition (6) in Lemma 2.1 is quite strong. It is seen at once that the only possible BSN that verifies it — and thus provides **unique** expansions for positive integers — is the classic binary system, $S = \langle 2^i \rangle$. This is the reason why we will relax condition (b) and accept a certain degree of redundancy in our BSN. Instead of a unique expansion, we must put up with an "almost unique" expansion.

Taking into account that we also have to fulfill (5), a relaxing condition for (6) is the following:

$$\forall i \in \mathbb{Z}^+ \quad L_{i+1} - L_i = L_{i-j}, \quad \text{for } j \in \{0, 1, 2, \ldots, i\}. \tag{10}$$

If we make $j = 0$ in (10) we obtain the recurrence

$$L_{i+1} = 2L_i,$$

which, together with $L_0 = 1$ determines completely $L_i = 2^i$. The BSN we obtain is the usual binary or dyadic system.

Making $j = 1$, we get the Fibonacci recurrence:

$$L_{i+1} = L_i + L_{i-1}.$$

And, as we mentioned before, the fact that $L_0 = 1$ and condition (a) imply that $L_1 = 2$. Thus, the base of the system is completely determined becoming the classical Fibonacci sequence (right-shifted):

$$F_0 = 1, \quad F_1 = 2, \quad F_2 = 3, \quad F_3 = 5, \quad F_4 = 8, \quad F_5 = 13, \quad F_6 = 21, \quad \ldots.$$

This BSN is known as the *Fibonacci binary system* or Zeckendorf's system, [18] or [19, pp. 281–282]. In this system, any number has redundant expansions in terms of 0's and 1's. For instance:

$$26 = F_6 + F_3 = 1001000;$$

$$26 = F_5 + F_4 + F_3 = 111000;$$

$$26 = F_6 + F_2 + F_1 = 1000110.$$

The redundancy in the Fibonacci system comes directly from the recurrence: any number can be written as different sums of F_i using $F_{i+1} = F_i + F_{i-1}$. Nevertheless among those expansions, only one is the one provided by the algorithm (in the example, $26 = F_6 + F_3 = 1001000$).

In the Fibonacci system, the algorithm will never output an expansion with a block 11 in it. Thus, the expansion will be **unique** provided no two 1's are consecutive. These expansions will be called *admissible*. This facilitates the recognition of a valid expansion under the Fibonacci system:

Criterion of admissibility. *In the Fibonacci BSN, any positive integer has a unique expansion as a sum of F_i where no two consecutive terms ever appear. Reciprocally, any finite sum of F_i with no two consecutive terms present is admissible and represents a definite integer.*

In general, condition (10) ensures that algorithm (4) never produces an expansion with a block of the form $100 \overset{j-1)}{\ldots} 01$. Instead, the algorithm will provide the block $1000 \overset{j+1)}{\ldots} 00$ and these will be the admissible expansions.

Remark. We can stretch a little bit the Fibonacci recurrence and we would get what are called *higher order Fibonacci systems*. In those systems $S = \langle L_i \rangle$ is the sequence of the Fibonacci numbers of order m ($m \geqslant 3$) defined recurrently as

$$L_n = L_{n-1} + L_{n-2} + \cdots + L_{n-m} \quad \text{for } n \geqslant m$$

with

$$L_0 = 1; \quad L_1 = 2; \quad L_2 = 2^2; \quad L_3 = 2^3; \quad \ldots; \quad L_{m-1} = 2^{m-1}.$$

In the BSN that we obtain, any positive integer has a unique expansion as a sum of L_i where there is no run of m consecutive 1.

As you see, there are an infinity of binary systems of numeration!

3. Complete BSNs

So far we have limited ourselves to deal with systems of numeration for positive integers. Let us see if we can "connect" in a natural way the BSN above with suitable systems of representation of real numbers in $(0, 1)$ allowing us to have a *complete binary system of numeration*, CBSN. A positive real number will then be represented positionally as a sequence of 0 and 1 where the integer part and the fractional part are separated, as usual, by a period.

The classic dyadic system is the perfect example of a CBSN for if we want to be able to apply the system of numeration

$$L_0 = 1; \quad L_1 = 2; \quad L_2 = 2^2; \quad \ldots$$

to a real number in $(0, 1)$, there is no problem extending the recurrence to the left of $L_0 = 1$

$$L_{-1} = 2^{-1}; \quad L_{-2} = 2^{-2}; \quad L_{-3} = 2^{-3}; \quad \ldots$$

and we get the two-way sequence $L_j = 2^j$, $j \in \mathbb{Z}$. This allows us to replace N in (4) by any $x \in \mathbb{R}^+$. The residues, N_{i_k} are then real numbers and the algorithm terminates or not depending on the character of x. So, the greedy algorithm applies exactly in the same way to the integer part of x, $[x]$, and to its fractional part, $x - [x]$.

A real number $x > 0$ is expanded as

$$x = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_j} + \frac{1}{2^{k_1}} + \frac{1}{2^{k_2}} + \cdots \quad \text{with} \begin{cases} i_1 > 1_2 > \cdots > i_j \geqslant 0 \\ \text{and} \\ 1 \leqslant k_1 < k_2 < \cdots. \end{cases}$$

We thus have what we have called a *complete system for real number representation*.

Can we do the same with the Fibonacci system?

4. The Fibonacci binary system

Let us highlight a basic property of the Fibonacci BSN:

$$\begin{cases} F_{2k} - 1 = F_{2k-1} + F_{2k-3} + \cdots + F_3 + F_1, \\ F_{2k+1} - 1 = F_{2k} + F_{2k-2} + \cdots + F_2 + F_0, \end{cases}$$

thus, for any given n,

$$F_n - 1 = F_{n-1} + F_{n-3} + \cdots.$$

So, the greatest integer that can be represented in the Fibonacci system using

$$F_0, F_1, F_2, \dots, F_{n-1}$$

is $F_n - 1 = F_{n-1} + F_{n-3} + F_{n-5} + \cdots$. Since any number less or equal than $F_n - 1$ has an expansion using these terms, there are exactly $F_n - 1$ admissible expansions that can be made using $F_0, F_1, F_2, \dots, F_{n-1}$.

If our aim is to obtain a CBSN, as the usual binary system, we must try to expand any real number $0 < x < 1$ as a series involving a Fibonacci sequence. As we did for the usual dyadic system, our first impulse would be to extend the classical Fibonacci sequence to "the left" of its first term, $F_0 = 1$ using negative indexes and keeping the recurrence valid. We would have necessarily:

$$F_{-1} = 1, \quad F_{-2} = 0, \quad F_{-3} = 1, \quad F_{-4} = -1, \quad F_{-5} = 2, \quad \dots.$$

There is no need to go any farther. As the F_i are integers the scheme, obviously, fails. This is so for two reasons: we have surpassed the 0, left endpoint of our reference set, $\mathbb{R}^+$, and besides, the sequence is alternating and not decreasing!

Having gone over the first impulse, a little reflection tells us that what we need is an extended sequence F_{-n} such that is both decreasing and having 0 as a limit point. The question then is: what value must F_{-1} have in order to fulfill these two conditions while, at the same time, satisfying the Fibonacci recurrence?

Let $F_{-1} = \lambda$, with $\lambda \in (0, 1)$. Using the Fibonacci recurrence backwards we would obtain:

$$F_{-1} = \lambda, \quad F_{-2} = 1 - \lambda, \quad F_{-3} = -1 + 2\lambda, \quad F_{-4} = 2 - 3\lambda, \quad F_{-5} = -3 + 5\lambda,$$

$$\dots$$

$$F_{-n} = \begin{cases} -F_{n-3} + F_{n-2} \cdot \lambda, & \text{for odd } n > 1, \\ F_{n-3} - F_{n-2} \cdot \lambda, & \text{for even } n > 2. \end{cases}$$

Let us now impose the condition that the F_{-n} be decreasing. This leads immediately to

$$\lambda < 1; \quad \lambda > \frac{1}{2}; \quad \lambda < \frac{2}{3}; \quad \lambda > \frac{3}{5}; \quad \dots.$$

We see at once that λ verifies

$$\frac{F_{2n}}{F_{2n+1}} < \lambda < \frac{F_{2n+1}}{F_{2n+2}}.$$

These are precisely the convergents of the regular continued fraction expansion [20, pp. 11–13]:

$$\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}},$$

which is no other than the expansion of the reciprocal of the golden number:

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

We have then,

$$\lambda = \frac{1}{\phi} = \frac{\sqrt{5} - 1}{2}.$$

Now, if $\lambda = \phi^{-1}$, and using the usual characterization of Fibonacci numbers adapted to our sequence,

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^{n+2} - (-\phi)^{-n-2}\right),$$

it is easy to see that our desired continuation of F_i to the left of 1 is precisely

$$F_{-n} = \phi^{-n},$$

and, obviously,

$$\lim_{n\to\infty} F_{-n} = 0.$$

We have, then, that for real numbers $0 < x < 1$, the "natural" extension of the Fibonacci system of numeration is the sequence

$$\frac{1}{\phi}, \frac{1}{\phi^2}, \frac{1}{\phi^3}, \ldots.$$

In the next subsection we see that what we have found is actually a system of representation.

4.1. The Fibonacci system of representation

In fact, what we are going to see is that any real number $0 < x < 1$ can be uniquely represented as a series of the form:

$$x = \sum_{i=1}^{\infty} \frac{1}{\phi^{s_i}}, \quad \text{where } s_1 \geqslant 1 \text{ and } s_i < s_i + 1 < s_{i+1}. \tag{11}$$

This can be proved just by noticing that (11) is actually a $(\tau, \tau - 1)$-expansion with $\tau = \phi$, see [9].

We are interested, though, in showing that the same greedy algorithm we used for the L_i system of numeration allows us to represent a real number in the way (11).

Let us consider the partition of the unit interval

$$[0, 1) = \bigcup_{i=1}^{\infty} \left[\frac{1}{\phi^i}, \frac{1}{\phi^{i-1}}\right).$$

Any real number $x \in [0, 1)$ must belong to one of the intervals:

$$x \in \left[\frac{1}{\phi^{s_1}}, \frac{1}{\phi^{s_1-1}}\right).$$

Consequently, we can write $x = x_1$ as:

$$x = \frac{1}{\phi^{s_1}} + \left(\frac{1}{\phi^{s_1-1}} - \frac{1}{\phi^{s_1}}\right) \cdot x_2 = \frac{1}{\phi^{s_1}} + \frac{1}{\phi^{s_1}}(\phi - 1)x_2 = \frac{1}{\phi^{s_1}} + \frac{1}{\phi^{s_1+1}} \cdot x_2,$$

with $x_2 \in (0, 1)$. Iterating the process with x_2, we end up with the desired expansion (11) which can also be written as:

$$x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{\phi^i}, \tag{12}$$

with $\varepsilon_i = 0, 1$ according to the absence or presence of ϕ^{-s_i} in (11) and the requirement that no two consecutive ε_i being equal to one. Expansion (12) allows us to view the expansion as a β-expansion [4,5], see Remark 2 below.

This leads immediately to a positional representation:

$$x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots, \quad \text{block 11 forbidden}.$$

It goes without saying that, as is the case in all additive systems of representation, any finite expansion has a non-terminating equivalent [21,22]. This is a consequence of the equation

$$\frac{1}{\phi^j} = \frac{1}{\phi^{j+1}} + \frac{1}{\phi^{j+3}} + \cdots + \frac{1}{\phi^{j+2n+1}} + \cdots. \tag{13}$$

Thus, the claimed uniqueness has to be excepted in the case of a finite expansion unless we agree we always choose the non-terminating case. These other redundancies in additive systems of real numbers representation cannot be avoided as [21,22] prove.

As a result we have a new CBSN: *the Fibonacci system*.

Remark 1. We mentioned before the *higher order Fibonacci systems*, where we use as $S = \langle L_i \rangle$ the Fibonacci numbers of order m ($m \geqslant 3$) defined recurrently as

$$L_n = L_{n-1} + L_{n-2} + \cdots + L_{n-m} \quad \text{for } n \geqslant m$$

with

$$L_0 = 1; \quad L_1 = 2; \quad L_2 = 2^2; \quad L_3 = 2^3; \quad \ldots; \quad L_{m-1} = 2^{m-1}.$$

In the resulting BSN, any positive integer has a unique expansion as a sum of L_i where there is no run of m consecutive terms. If we now consider the number $1 < \rho < 2$ solution of

$$\rho^m = \rho^{m-1} + \rho^{m-2} + \cdots + \rho + 1, \tag{14}$$

we have

$$1 = \frac{1}{\rho^m} + \frac{1}{\rho^{m-1}} + \cdots + \frac{1}{\rho}$$

and the recurrence may be taken backwards from L_0:

$$L_{-1} = \frac{1}{\rho}; \quad L_{-2} = \frac{1}{\rho^2}; \quad \ldots; \quad L_{-m} = \frac{1}{\rho^m}; \quad \ldots.$$

Remark 2. Alternatively, if we do not demand the L_i to be integers, we have quite a new world of possibilities. Let $1 < \beta < 2$, and let

$$L_0 = 1; \quad L_1 = \beta; \quad L_2 = \beta^2; \quad L_3 = \beta^3; \quad L_4 = \beta^4; \quad \ldots.$$

Notice that the recurrence here is $L_n = \beta \cdot L_{n-1}$. Algorithm (4) does not terminate for a starting integer N until you get a residue between 0 and 1. Then, continuing the series for negative indexes:

$$L_{-1} = \frac{1}{\beta}; \quad L_{-2} = \frac{1}{\beta^2}; \quad L_{-3} = \frac{1}{\beta^3}; \quad \ldots,$$

we obtain a complete representation BSN called the β-adic system. The inconvenient of this system is that for many values of β, positive integers have necessarily decimal expansions. Except for that, the system is very interesting.

The series

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{\beta^i}, \quad \varepsilon_i = 0, 1,$$

for real number representation in $(0, 1)$ are called *β-expansions*, see [4,5].

Among β-expansions, particularly important are those where β is an algebraic integer and, specially, the cases where β is a Pisot or a Salem number. See [23, pp. 53–56] for details. The number ρ above, the positive root of Eq. (14), is a Pisot number.

The particular case for $\beta = \phi = \frac{1+\sqrt{5}}{2}$, the golden mean verifying the equation $\beta^2 - \beta - 1 = 0$, is known as the ϕ-system and was described for the first time by a young 12-year-old student, George Bergman [24].

5. Van der Corput's sequence for the Fibonacci system

Van der Corput's sequence (2) is a uniformly distributed sequence of real numbers in $(0, 1)$ with the lowest known discrepancy. We recall that a sequence $\langle x_n \rangle$ in the unit interval $[0, 1]$ is uniformly distributed if for all α, β $(0 \leqslant \alpha < \beta < 1)$

$$\lim_{n\to\infty} \frac{A([\alpha, \beta); n)}{n} = \beta - \alpha, \tag{15}$$

where $A([\alpha, \beta); n)$ is the counting function of the number of x_i among the first n contained in the interval $[\alpha, \beta)$:

$$A\big([\alpha, \beta); n\big) = \#\big\{i\colon\ x_i \in [\alpha, \beta);\ 1 \leqslant i \leqslant n\big\}.$$

The discrepancy of a finite sequence $\langle x_1, \ldots, x_n \rangle$ in $[0, 1]$ is the value

$$D_n = \sup_{0 \leqslant \alpha < \beta \leqslant 1} \left| \frac{A([\alpha, \beta); n)}{n} - (\beta - \alpha) \right|.$$

The discrepancy $\langle D_n \rangle$ of an infinite sequence, is the sequence of the discrepancies of its first n terms. For a uniformly distributed sequence, the discrepancy is used to measure the "degree" of adjustment of the sequence to the ideal uniform distribution. A sequence with low discrepancy is a sequence with a "good" uniform distribution. For a thorough treatment, we recommend the excellent book [25] or the more recent [26]. A known result is the following [25, p. 89]:

Theorem 5.1. *The sequence $\langle x_n \rangle$ in $(0, 1)$ is uniformly distributed if and only if* $\lim_{n\to\infty} D_n = 0$.

The convergence to 0 of the discrepancy $\langle D_n \rangle$ of a uniformly distributed sequence cannot be too rapid as for any sequence it is easily proved that

$$\frac{1}{n} \leqslant D_n \leqslant 1. \tag{16}$$

In fact, this bound can be improved. Schmidt proved [27] that there is a constant $c > 0$ such that infinitely many times verifies:

$$c\frac{\log n}{n} \leqslant D_n \leqslant 1. \tag{17}$$

Van der Corput's sequence (2) was exhibited as a uniformly distributed sequence in $(0, 1)$ with an extremely small discrepancy. Formally, the sequence, $\langle x_i \rangle$, is the following:

Definition 1 *(Van der Corput sequence).*

$$\text{If } n = \sum_{i=1}^{m} 2^{r_i}; \quad 0 \leqslant r_1 < r_2 < \cdots < r_m, \quad \text{then } x_n = \sum_{i=1}^{m} \frac{1}{2^{r_i+1}}. \tag{18}$$

The discrepancy of a sequence is not easy to establish. In the case of Van der Corput's sequence, its discrepancy satisfies [25, p. 127]

$$D_n \leqslant \frac{\log(n+1)}{n \log 2}, \tag{19}$$

or, even better [28]

$$D_n \leqslant \frac{1}{3\log 2}\frac{\log n}{n} + \frac{1}{n}. \tag{20}$$

This last is a rather tight bound if we take into account (16) and (17). The constant $1/(3\log 2)$ is best possible.

It is obvious that the construction of Van der Corput's sequence takes direct advantage of the fact that the usual dyadic system is a CBSN. Let us see what happens if we apply the same idea to our complete Fibonacci system,

$$F_0 = 1; \quad F_1 = 2; \quad F_{i+1} = F_i + F_{i-1} \quad \forall i \geqslant 1.$$

Definition 2 *(Fibonacci–Van der Corput sequence).*

$$\text{If } n = \sum_{i=1}^{m} F_{r_i} \quad (0 \leqslant r_1 < r_2 < \cdots < r_m,\ r_{i+1} > r_i + 1) \quad \text{then } x_n = \sum_{i=1}^{m} \frac{1}{\phi^{r_i+1}}. \tag{21}$$

As in the case of the original Van der Corput's sequence, we can prove the following result:

Theorem 5.2. *The Fibonacci–Van der Corput sequence* (21) *is uniformly distributed in* $(0, 1)$.

In order to prove this result, we offer three lemmas that will be useful in the sequel (the reader will have no difficulty in proving them. The first one can be tackled by induction on t).

Lemma 5.3. *Given $j, t \in \mathbb{Z}$, $0 \leqslant j < t$, the number of expressions of the form*

$$\sum_{i=1}^{m} \frac{1}{\phi^{r_i+1}}; \quad j \leqslant r_1 < \cdots < r_m < t;\ r_i + 1 < r_{i+1},$$

is exactly $F_{t-j} - 1$.

Lemma 5.4. *Given $j \in \mathbb{Z}^+$ we have*

$$\lim_{t\to\infty} \frac{F_{t-j} - 1}{F_t - 1} = \frac{1}{\phi^j}.$$

Lemma 5.5. *If we have m sequences of real numbers, formed by fractions of positive numerator and denominator*

$$\left\langle \frac{a_i^j}{b_i^j} \right\rangle_i, \quad j = 1, \ldots, m,$$

such that all have the same limit, k, we have

$$\lim_{i\to\infty} \frac{\sum_{j=1}^m a_i^j}{\sum_{j=1}^m b_i^j} = k.$$

We are now ready to offer the proof of the uniform distribution of the sequence (21).

It is not difficult to see that definition (15) is equivalent to:

$$\text{for any } 0 \leqslant s \leqslant 1, \quad \lim_{n\to\infty} \left| \frac{A([0,s);n)}{n} \right| = s.$$

Thus, we have to prove that for any $s \in [0,1]$ we have

$$\lim_{n\to\infty} \frac{\#\{i\colon x_i < s;\ 1 \leqslant i \leqslant n\}}{n} = s.$$

To start with, we consider the following special values for $s \in [0,1]$ and $n \in \mathbb{Z}^+$:

$$s = \frac{1}{\phi^{a_1}} + \cdots + \frac{1}{\phi^{a_k}}; \quad 1 \leqslant a_1 < \cdots < a_k; \quad \text{and} \quad n = F_t - 1.$$

Let us find the exact number of terms of the sequence x_i, with $i < F_t$ that comply with the condition $x_i < s$. In order to do that, we will partition the interval $[0,s)$ in k disjoint subintervals:

$$[0,s) = \left[0, \frac{1}{\phi^{a_1}}\right) \cup \left[\frac{1}{\phi^{a_1}}, \frac{1}{\phi^{a_1}} + \frac{1}{\phi^{a_2}}\right) \cup \cdots \cup \left[\sum_{i=1}^{k-1} \frac{1}{\phi^{a_i}}, s\right) = \bigcup_{j=1}^{k} I_j.$$

We have,

$$\#\{i\colon x_i < s;\ 1 \leqslant i \leqslant n\} = \sum_{j=1}^{k} \#\{i\colon x_i \in I_j;\ 1 \leqslant i \leqslant n\} = \sum_{j=1}^{k} (F_{t-a_j} - 1).$$

This last equality is a direct consequence of Lemma 5.3 (with $j = 0$), applied to each one of the k intervals of the partition. We must remark that in the case that index t be less than or equal to the a_j, we would formally have $F_{t-a_j} - 1 = 0$, which is the same than saying that in these cases there are no terms of the sequence x_i in the corresponding intervals I_j. Finally, if we find the limit of the frequency of the $x_i < s$ we have

$$\lim_{t\to\infty} \frac{\sum_{j=1}^k (F_{t-a_j} - 1)}{F_t - 1} = \lim_{t\to\infty} \sum_{j=1}^{k} \frac{F_{t-a_j} - 1}{F_t - 1} = \sum_{j=1}^{k} \frac{1}{\phi^{a_j}} = s.$$

The last equality is a direct consequence of Lemma 5.4.

We now remake the above calculations for a general n:

$$n = F_{b_1} + F_{b_2} + \cdots + F_{b_r}, \quad b_1 > b_2 > \cdots > b_r.$$

The count of the $x_i < s$ up to index $i = F_{b_1} - 1$ has been carried out above with the result $\sum_{j=1}^k (F_{b_1 - a_j} - 1)$.

Let us extend the count to those indexes between F_{b_1} and $F_{b_1} + F_{b_2} - 1$. These values of n are of the form:

$$F_{b_1} + \sum_{j=s_1}^{s_l} F_j, \quad 0 \leqslant s_1 < s_2 < \cdots < s_l < b_2,\ s_i + 1 < s_{i+1}.$$

The terms of the sequence that correspond to those indexes are of the form:

$$x_n = \frac{1}{\phi^{b_1+1}} + \sum_{j=s_1}^{s_l} \frac{1}{\phi^{b_j+1}},$$

and among these terms, those that are less than s will be

$$\sum_{j=1}^{k} (F_{b_2 - a_j} - 1).$$

Proceeding in the same way we will have the final count of the terms of the sequence less than s:

$$\sum_{i=1}^{r}\left(\sum_{j=1}^{k}(F_{b_i-a_j}-1)\right).$$

If we allow n to tend to ∞ we face the following limit:

$$\lim_{b_r\to\infty}\frac{\sum_{i=1}^{r}(\sum_{j=1}^{k}(F_{b_i-a_j}-1))}{\sum_{i=1}^{r}F_{b_i}}=s.$$

Using Lemma 5.5, the limit is s for all those indexes less than b_r that occasionally may tend also to ∞ alongside with b_r.

As for remaking the whole count for an arbitrary s, it is quite obvious that any $s\in[0,1]$ can be approached as closely as we wish using finite expressions as

$$s=\frac{1}{\phi^{a_1}}+\cdots+\frac{1}{\phi^{a_k}},\quad 1\leqslant a_1<\cdots<a_k.$$

Consequently, our proof extends, by continuity, to any value of s and we are done.

At this point, a natural question to ask ourselves is about the "fitness" of our sequence to the ideal uniform distribution. Can we obtain a similar result to (19) or (20)? We conjecture that the answer is positive and that in the case of our Fibonacci–Van der Corput's sequence, the discrepancy verifies

$$D_n\leqslant\frac{\log(n+1)}{n\log\phi},$$

but we have not been able to prove it yet.

6. Metrical properties of the Fibonacci system

The usual binary system for real number representation has some very interesting properties concerning the distribution of digits in a non-terminating expansion. The most remarkable is the property, discovered by Émile Borel, that for *almost all* real numbers in $(0,1)$, the digits 0 and 1 appear approximately with the same frequency. The *almost all*, as usual, means for all numbers in $(0,1)$ except for a set of Lebesgue measure zero. To be more precise, if x has the dyadic expression,

$$x=0.\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots,\quad \varepsilon_i\in\{0,1\},$$

then

$$\lim_{n\to\infty}\frac{\#\{i:\ \varepsilon_i=0;\ 1\leqslant i\leqslant n\}}{n}=\lim_{n\to\infty}\frac{\#\{i:\ \varepsilon_i=1;\ 1\leqslant i\leqslant n\}}{n}=\frac{1}{2}. \tag{22}$$

If the same x is written as

$$x=\sum_{i=1}^{\infty}\frac{1}{2^{s_i}},\quad 1\leqslant s_1<s_2<\cdots,$$

(22) is equivalent to

$$\lim_{n\to\infty}\frac{s_n}{n}=2$$

as s_n/n is the reciprocal of expression (22) in what concerns the number of 1's.

Borel defined such numbers as *simply normal* to base 2. If, besides containing digits 0 and 1 in the due proportions, any other block of digits, $\varepsilon_1\varepsilon_2\ldots\varepsilon_k$, of length k appears with frequency $1/2^k$, the number was defined as *normal* to base 2.

The function $T(0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots)=0.\varepsilon_2\varepsilon_3\varepsilon_4\ldots$ is called the *right-shift function*, or simply the *shift function* and allows us to interpret the iterative algorithm that leads to the expansion of a real number in $(0,1)$ as a dynamical system. The graph of the binary shift function (see Fig. 1(a)) is a well-known graph for a beginner student of dynamical systems and iterative functions. As we mentioned before, the Fibonacci system of representation is a particular case for $\tau=\phi$ of what we called in [9] a $(\tau,\tau-1)$-expansion. These expansions can also be considered as particular cases of generalized Lüroth series (GLS) [23, pp. 41–48]. For the sake of completeness, we recall the main details of $(\tau,\tau-1)$-expansions as given in [9].

Definition 3 *($(\tau,\tau-1)$-expansions).* Given $\tau\in\mathbb{R}$, $\tau>1$, any real number $x\in[0,1)$ has the following (unique except for finite sums) expansion:

$$x=\sum_{i=1}^{\infty}\varepsilon_i\frac{(\tau-1)^{\sum_{j=1}^{i-1}\varepsilon_j}}{\tau^i},\quad \varepsilon_i\in\{0,1\}. \tag{23}$$

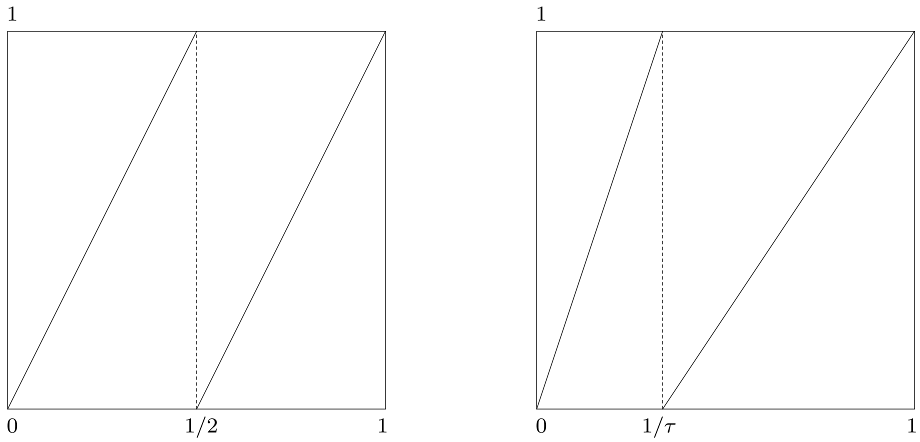

Fig. 1. (a) Dyadic and (b) $(\tau, \tau - 1)$ right-shifts.

Thus, every $x \in [0, 1)$ can be represented through its 'digits', $\langle \varepsilon_i \rangle$. The right-shift, T, induced by expansion (23) on $[0, 1)$,

$$T(0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots) = 0.\varepsilon_2\varepsilon_3\varepsilon_4\ldots,$$

has the graph shown in Fig. 1(b). T leaves invariant Lebesgue measure and is ergodic with respect to it [23, pp. 41–48]. This result, via the ergodic theorem, is equivalent to saying that for almost all $x \in (0, 1)$ the orbit $\langle T^n x \rangle$, is uniformly distributed in $(0, 1)$.

Expansion (23) has a compact form [9, p. 594]:

$$x = \sum_{i=1}^{\infty} \frac{(\tau - 1)^{i-1}}{\tau^{s_i}}, \quad 1 \leqslant s_1 < s_2 < s_3 < \cdots.$$

In the case $\tau = 2$ we obtain the usual dyadic system and for $\tau = \phi$, as $\phi - 1 = 1/\phi$, we have

$$x = \sum_{i=1}^{\infty} \frac{1}{\phi^{s_i} \cdot \phi^{i-1}} = \sum_{i=1}^{\infty} \frac{1}{\phi^{s_i+i-1}}, \quad 1 \leqslant s_1 < s_2 < s_3 < \cdots. \tag{24}$$

If we write $k_i = s_i + i - 1$, (24) can also be written

$$x = \sum_{i=1}^{\infty} \frac{1}{\phi^{k_i}}, \quad k_1 \geqslant 1,\ k_{i+1} > k_i + 1. \tag{25}$$

We find again an expansion that coincides with a β-expansion for $\beta = \phi$ as studied by Rényi [4] and Parry [5].

Borrowing Borel's terminology, we will define:

Definition 4 *(Normal number to the Fibonacci system).* A real number $x \in [0, 1)$ is *normal* to the Fibonacci system if its orbit $\langle T^n x \rangle$ is uniformly distributed in $[0, 1)$.

Normal numbers to the Fibonacci system in $[0, 1)$ constitute, as we have mentioned above, a set of measure one. With this definition and both (24) and (25) in mind, the following theorem can be proved [9, p. 594]:

Theorem 6.1. *Let*

$$x = \sum_{i=1}^{\infty} \frac{1}{\phi^{k_i}}, \quad k_1 \geqslant 1,\ k_{i+1} > k_i + 1,$$

be a normal number to the Fibonacci system. Then,

$$\lim_{n\to\infty} \frac{k_n}{n} = \lim_{n\to\infty} \frac{s_n + n - 1}{n} = \frac{5 + \sqrt{5}}{2}.$$

This result can also be obtained considering (25) as a β-expansion, see [23, p. 78].

7. A continuous, strictly increasing singular function

We are now going to exhibit a function $L:[0,1]\to[0,1]$, continuous, strictly increasing, and singular whose definition derives directly from the two CBSN we have been using: the usual dyadic system and the Fibonacci system.

Singular functions are those whose derivative vanish almost everywhere on their domain. Continuous, non-decreasing singular functions are, to say the least of them, peculiar. The best known among them is Cantor–Lebesgue function which is constant on each of the intervals that form Cantor's set complement.

But **strictly** increasing singular functions are even more peculiar. There are no intervals of constancy and yet the derivative vanishes almost everywhere on $(0,1)$. The best known examples of these are Minkowski's *fragefunktion*, $?(x)$ [29–31], and the Riesz–Nágy function [16, pp. 48–49].

Let us define $L:[0,1]\to[0,1]$ in the following way.

Definition 5 *(The singular function L).* Let $x\in[0,1]$ be written in the Fibonacci system,

$$x=\sum_{i=1}^{\infty}\frac{1}{\phi^{k_i}},\quad k_1\geqslant 1,\ k_{i+1}>k_i+1.$$

We recover its $(\phi,\phi-1)$-expansion form by changing $k_i=s_i-i+1$:

$$x=\sum_{i=1}^{\infty}\frac{1}{\phi^{s_i+i-1}},\quad 1\leqslant s_1<s_2<s_3<\cdots,\tag{26}$$

and now define $L(x)$ as the dyadic number:

$$L(x)=\sum_{i=1}^{\infty}\frac{1}{2^{s_i}}.\tag{27}$$

It is quite easy to see that function L just defined is continuous and strictly increasing on $[0,1]$. In fact, it is function $\Phi_{\phi,2}$ of the generalized Riesz–Nágy–Takács family, $\Phi_{\alpha,\tau}$, we defined in [9]. The singularity of these functions was proved there in Theorem 4.2 which, adapted to our function $L(x)=\Phi_{\phi,2}$ says:

Theorem 7.1. *Let*

$$K=\frac{\log\phi}{\log(\phi/2)}\approx 2.2705\ldots$$

and let $x\in[0,1)$ have the $(\phi,\phi-1)$-expansion (26)*. If $L'(x)$ exists* (*in a wide sense*) *and there exists a value $k\in\mathbb{R}$ such that*:

(a) $\liminf_{n\to\infty}\frac{s_n}{n}\geqslant k>K$ *then* $L'(x)=0$;
(b) $\limsup_{n\to\infty}\frac{s_n}{n}\leqslant k<K$ *then* $L'(x)=\infty$.

Now, as we have seen before in Theorem 6.1, all the x in the set of normal numbers to the Fibonacci system — or what is the same, to the $(\phi,\phi-1)$-expansion (26) — N_ϕ, verify

$$\lim_{n\to\infty}\frac{s_n}{n}=\frac{3+\sqrt{5}}{2}=1+\phi.$$

Thus, according to Theorem 7.1, the set N_ϕ (of measure one) lies entirely in the region where $L'=0$. This proves the singularity of L.

Incidentally, the same theorem allows us the identify a set of numbers (necessarily of measure zero) for whom $L'=\infty$: the inverse image of the set of normal dyadic numbers: N_2. This set consists of numbers for whom it can verified

$$\lim_{n\to\infty}\frac{s_n}{n}=2.$$

Consequently, the numbers in $L^{-1}(N_2)$ lie entirely in the region where $L'=\infty$. The situation can be quickly grasped graphically in Fig. 2.

As a last touch of the peculiar character of singular functions let us remark that if we transform our Fibonacci–Van der Corput sequence (21) using function L, the new sequence has an asymptotic distribution function that is exactly L^{-1}, that is, function $\Phi_{2,\phi}$ of the Riesz–Nágy–Takács functions.

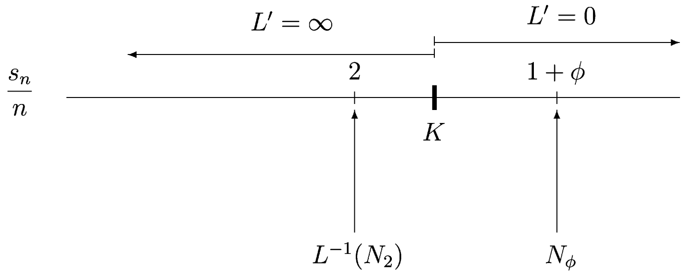

Fig. 2. The value of $L'(x)$ according to the position of s_n/n.

7.1. A direct proof of the singularity of $L(x)$

The previous proof of the singularity of L was based on a general result obtained for a family of functions. In the case of our function L, its definition based on the two CBSN allows us a direct proof of its singular character.

Let

$$x = [s_1, s_2, \ldots, s_n, \ldots] = \sum_{i=1}^{\infty} \frac{1}{\phi^{s_i+i-1}}, \tag{28}$$

where, as usual, we require that the s_i be an increasing sequence of positive integers. For reasons that will be clear presently, let us also assume that x does not have consecutive s_n from some point onwards. The set of these numbers is numerable and, consequently, of measure zero. That makes it irrelevant from the point of view of our purpose.

Let $x_n = [s_1, s_2, \ldots, s_n]$ be the n-th partial sum of the series:

$$x_n = [s_1, s_2, \ldots, s_n] = \sum_{i=1}^{n} \frac{1}{\phi^{s_i+i-1}}, \quad \text{where } 1 \leqslant s_1 < s_2 < \cdots < s_n. \tag{29}$$

Clearly $x_n < x$. Assuming that $s_{n+1} > s_n + 1$, we now consider

$$y_n = [s_1, s_2, \ldots, s_n, s_n + 1] = \sum_{i=1}^{n} \frac{1}{\phi^{s_i+i-1}} + \frac{1}{\phi^{s_n+1+n}}.$$

We have clearly $x < y_n$. L is a strictly increasing function, hence

$$x_n < x < y_n \quad \Rightarrow \quad L(x_n) < L(x) < L(y_n).$$

Now, $L(x_n) = \sum_{i=1}^{n} \frac{1}{2^{s_i}}$ and thanks to the way we have chosen y_n, $L(y_n) = \sum_{i=1}^{n} \frac{1}{2^{s_i}} + \frac{1}{2^{s_n+1}}$.

Since both $x_n \to x$ and $y_n \to x$, if $L'(x)$ exists, it has to coincide with the limit

$$\begin{aligned} L'(x) &= \lim_{n\to\infty} \frac{L(y_n) - L(x_n)}{y_n - x_n} \\ &= \lim_{n\to\infty} \frac{1/2^{s_n+1}}{1/\phi^{s_n+1+n}} \\ &= \frac{\phi}{2} \cdot \lim_{n\to\infty} \frac{\phi^{s_n+n}}{2^{s_n}} \\ &= \frac{\phi}{2} \cdot \lim_{n\to\infty} \left[\left(\frac{\phi}{2}\right)^{s_n/n} \cdot \phi \right]^n. \end{aligned}$$

Let us now consider the sequence involved under this last limit and denote it δ_n:

$$\delta_n = \left[\left(\frac{\phi}{2}\right)^{s_n/n} \cdot \phi \right]^n.$$

If the limit exists and is finite and different from 0 then, forcedly,

$$\lim_{n\to\infty} \frac{\delta_{n+1}}{\delta_n} = 1.$$

But

$$\lim_{n\to\infty} \frac{[(\frac{\phi}{2})^{s_{n+1}/(n+1)} \cdot \phi]^{n+1}}{[(\frac{\phi}{2})^{s_n/n} \cdot \phi]^n} = \phi \cdot \lim_{n\to\infty} \left(\frac{\phi}{2}\right)^{s_{n+1}-s_n}$$

and this last limit cannot be 1 since, for all n, $s_{n+1} - s_n$ is a positive integer. As an increasing function must have a finite derivative almost everywhere, the singularity of L is proved.

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